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Optimal control of the rectilinear motion of a rigid body on a rough plane by means of the motion of two internal masses☆

N.N. Bolotnik, T.Yu. Figurina

Moscow, Russia

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ABSTRACT

The problem of the optimal control of a rigid body moving along a rough horizontal plane due to motion of two internal masses is solved. One of the masses moves horizontally parallel to the line of motion of the main body, while the other mass moves in the vertical direction. Such a mechanical system models a vibration-driven robot–a mobile device able to move in a resistive medium without special propellers (e.g., wheels, legs or caterpillars). Periodic motions are constructed for the internal masses to ensure velocity-periodic motion of the main body with maximum average velocity, provided that the period is fixed and the magnitudes of the accelerations of the internal masses relative to the main body do not exceed prescribed limits. Based on the optimal solution obtained for a fixed period without any constraints imposed on the amplitudes of vibration of the internal masses, a suboptimal solution that takes such constraints into account is constructed.

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Mobile mechanisms, which move as a result of the motion of internal masses, with the main body being in direct contact with the environment, have a number of advantages over conventional vehicles. They are simple in design and their bodies can be made hermetic, without protruding components. This enables such devices to be used in a severely restricted space (for example, in narrow tubes) and in "vulnerable" media, for example, inside a human body to deliver a drug or a diagnostic sensor to an affected area. Li, Furuta, and Chernousko (Ref. 1) have mentioned the possibility of using such microrobots for diagnosing diseases of the digestive tract. Such mechanisms can be fixed in a prescribed position with a high degree of accuracy ($\sim 10^{-8}$ m),[2] which enables them to be used in high-precision positioning units in electron and tunnel microscopes, as well as in micro- and nanotechnology equipment. A similar locomotion principle is apparently inherent in some living creatures that do not have extremities (e.g., worms or snakes) and move by redistributing mass along their bodies. Mechanical systems of this type can serve as models for verifying this conjecture and studying the features of motion specific to these creatures. Such models have been constructed and investigated by a number of authors (see, e.g., Refs. 3–6).

Sometimes, automatic vehicles, moving by means of the motion of internal masses are called *vibration-driven robots*, since in the basic operating mode, the internal masses of such systems usually perform periodic vibrations.

Chernousko (Refs. 7,8) was the first to formulate the problem of the optimal control of the motion of a body with movable internal masses. He considered the rectilinear motion of a rigid body with one movable internal mass along a rough horizontal plane. The internal mass was allowed to move within fixed limits along a line parallel to the line of motion of the main body. Coulomb friction was assumed to act between the plane and the body. Periodic control modes were constructed for the relative motion of the internal mass to provide velocity-periodic motion of the main body such that the body moves through the same distance in the prescribed direction in each period. It was assumed that, at the beginning and end of each period, the velocity of the main body is equal to zero and the internal mass is located in its extreme left-hand position and also has zero velocity. Velocity-controlled and acceleration-controlled modes of motion were considered for the internal mass. In the first case, the internal mass moves with constant velocity, different for motion in the desired direction of motion of the main body and for motion in the opposite direction. The second mode implies for each period three intervals in which the relative acceleration of the internal mass is constant. For both modes, the optimal parameters for which the average velocity of the main body is a maximum were determined.

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E-mail address: bolotnik@ipmnet.ru (N.N. Bolotnik).

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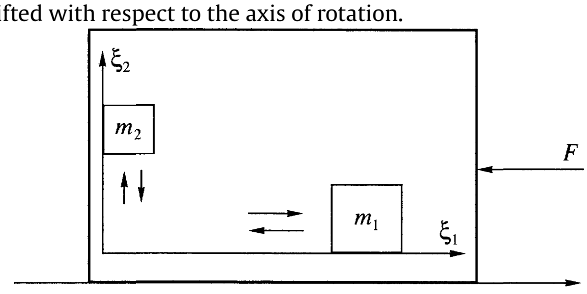
The average velocity of the steady motion sustained by periodic motion of the internal masses is one of the basic operating characteristics of vibration-driven robots, and the maximization of this velocity is an important problem for programming control modes for such systems.

Figurina (Ref. 9) has constructed an optimal control for the velocity-periodic motion of the mechanical system considered previously which maximizes the displacement of the main body for a fixed period and, hence, the average velocity over that period. The acceleration of the internal mass relative to the main body is used as the control variable. The magnitude of this acceleration is constrained. Without loss of generality, the velocity of the main body at the beginning and the end of the period is assumed to be equal to zero. The motion of the internal mass is subjected only to the periodicity condition. The constraint on the vibration amplitude for the internal mass, as well as the requirement for the relative velocity of the internal mass to be equal to zero at the beginning and end of the vibration period, when the velocity of the main body is zero, are not imposed. In addition, no constraints are imposed on the structure of the optimal control (e.g., bang-bang character or a specified number of switching instants). For such a statement, the average velocity of the main body of the system and the amplitude of relative vibrations of the internal mass turn out to be monitonically increasing functions of the vibration period and the maximum magnitude allowed for the relative acceleration of the internal mass. By changing the period, it is possible to provide a prescribed amplitude for the vibration of the internal mass. However, the question of the optimality (in the sense of the maximization of the average velocity) of the solution constructed in this way for the case when the period is free but the vibration amplitude of the internal mass is constrained remains open. The important fact has been established that by increasing the maximum magnitude allowed for the relative acceleration of the internal mass one can provide any prescribed average velocity of the main body for a fixed amplitude of vibration of the internal mass. This is not the case for control modes that prescribe that the velocity of the main body and the relative velocity of the internal mass should vanish simultaneously at the beginning and end of the period (Refs. 7,8).

The control of the motion of the internal masses of a vibration-driven robot produces a controlled change in the force of friction between the robot's body and the supporting surface, which enables the motion of the body to be controlled. The magnitude of the dry friction force acting on the body depends both on the sum of the other forces acting on the body in the direction of its motion and the normal reaction force of the supporting surface. If the internal masses move along the line of motion of the body, the normal reaction force does not change. Therefore, the normal reaction is uncontrolled in the systems described previously (Refs. 7–9). To enable the normal reaction to be controllable it is necessary for the internal masses to be allowed to move in a direction perpendicular to the supporting surface. Bolotnik et al. (Ref. 10) considered the rectilinear motion of a model of a vibration-driven robot with two internal masses along a rough horizontal plane. One of the internal masses moves relative to the body along its line of motion, whereas the other mass moves along the vertical, which makes it possible to influence the normal reaction of the supporting surface. Both masses perform harmonic vibrations that have the same frequency but are shifted in phase. It was shown that by controlling the phase shift and the frequency of the vibrations of the internal masses one can change the direction of motion of the body and the average velocity of the steady (velocity-periodic) motion of the robot. The optimal value of the phase shift, for which the magnitude of the average velocity is a maximum was found. An approximate expression for the average velocity of the steady motion was obtained for the case of a small coefficient of friction between the robot's body and the supporting plane. This expression enabled the dependence of this velocity on the parameters of the vibrations of the internal masses to be analyzed. It was established that in this case, the optimal phase shift between the vertical and horizontal vibrations is close to $\pi/2$. This fact justifies the use of an unblance vibration exciter as the actuator for vibration-driven robots. Such an exciter consists of a rotor, the center of mass of which is shifted with respect to the axis of rotation.



In the system considered below, the normal reaction of the surface is controlled by means of an independently driven internal mass that moves along the vertical. The special case of zero acceleration of this mass corresponds to the optimal control problem solved previously[9] for a system with one internal mass.

1. The mathematical model of the system and the statement of the optimal control problem

Consider a mechanical system consisting of a rigid body (the main body) of mass m_0 and two internal bodies (point masses) of mass m_1 and m_2 (see the figure). The main body moves translationally along a straight line on a rough horizontal plane. The internal bodies move relative to the main body in a vertical plane parallel to the line of motion of the main body. Body m_1 moves in a horizontal direction, while body m_2 moves along the vertical. There is dry (Coulomb) friction acting between the main body and the supporting plane. For the system described, we will solve the optimal control problem for periodic motions of the internal bodies to maximize the average velocity of the main body.

We introduce in the vertical plane, in which the internal bodies move, two right-handed rectangular coordinate frames—a fixed (inertial) frame Oxy and the coordinate frame $O'\xi_1\xi_2$, rigidly attached to the main body. The x- and ξ_1- axes are horizontal, while the y- and ξ_2-axes are directed vertically upward. The line OO' is horizontal. Without loss of generality we will assume that body m_1 moves along the ξ_1-axis and body m_2 moves along the ξ_2-axis. Let x denote the abscissa of the point O' in the coordinate frame Oxy (the displacement of the main body relative to the fixed reference frame), let ξ_1 and ξ_2 denote the abscissa of body m_1 and the ordinate of body m_2, respectively, in the coordinate frame $O'\xi_1\xi_2$, let $M = m_0 + m_1 + m_2$ be the total mass of the system, let k be the dry friction coefficient and let g be the acceleration due to gravity.

The motion of the main body along the x-axes for specified motions of the internal bodies is governed by the relations

$$M\ddot{x} + m_1\ddot{\xi}_1 = F \tag{1.1}$$

$$F = \begin{cases} -kN\,\mathrm{sign}\,\dot{x}, & \text{if } \dot{x} \neq 0 \\ m_1\ddot{\xi}_1, & \text{if } \dot{x} = 0 \text{ and } |m_1\ddot{\xi}_1| \le kN \\ kN\,\mathrm{sign}(m_1\ddot{\xi}_1), & \text{if } \dot{x} = 0 \text{ and } |m_1\ddot{\xi}_1| > kN \end{cases} \tag{1.2}$$

$$N = Mg + m_2\ddot{\xi}_2 \tag{1.3}$$

where F is the dry friction force that obeys Coulomb's law and N is the normal pressure force.

It is assumed that the main body is in permanent contact with the plane and, hence,

$$Mg + m_2\ddot{\xi}_2 \ge 0 \tag{1.4}$$

This inequality imposes a constraint on the admissible values of the relative acceleration of body m_2.

We will consider periodic motions of the internal bodies with a given period T that excite velocity-periodic motion of the main body, i.e.,

$$\xi_i(t+T) = \xi_i(t), \quad \dot{x}(t+T) = \dot{x}(t) \tag{1.5}$$

Due to the periodicity, it suffices to consider the motion in the interval $[0,\ T]$. Without loss of generality, we put

$$\xi_1(0) = 0, \quad \xi_2(0) = 0, \quad x(0) = 0, \quad \dot{x}(0) = 0 \tag{1.6}$$

These relations can be satisfied by an appropriate choice of the reference points for measuring the coordinates and time.

As regards the last condition of (1.6), note that the velocity of the main body $\dot{x}$ necessarily vanishes at some instant of time. Indeed, since $\dot{x}$ and ξ_i are periodic functions of time, $\ddot{x}$ and $\ddot{\xi}_i$ are periodic functions with zero means. According to the equation of motion (1.1), the friction force F is also a periodic function with zero mean and, hence, is alternating. Therefore, the velocity $\dot{x}$, which changes continuously with time, cannot be constant in sign and, hence, vanishes at some instant of time.

Using relations (1.5) and (1.6), we define the boundary conditions in the interval $[0,\ T]$:

$$\xi_i(0) = \xi_i(T) = 0, \quad \dot{\xi}_i(0) = \dot{\xi}_i(T), \quad i = 1, 2 \tag{1.7}$$

$$x(0) = 0, \quad \dot{x}(0) = \dot{x}(T) = 0 \tag{1.8}$$

We assume that due to the restricted powers of the drives, the magnitudes of the relative accelerations of the internal bodies, $\ddot{\xi}_1$ and $\ddot{\xi}_2$, do not exceed the values A_1 and A_2, respectively. Then, in view of inequality (1.4), we obtain the constraints

$$|\ddot{\xi}_1(t)| \le A_1, \quad -A_2^- \le \ddot{\xi}_2(t) \le A_2, \quad A_2^- = \min\left\{A_2, \frac{Mg}{m_2}\right\} \tag{1.9}$$

To enable the main body of the system to be moved from a state of rest, it is necessary to subject the parameters of the system to the condition

$$m_1A_1 > k(Mg - m_2A_2^-) \tag{1.10}$$

We introduce the dimensionless variables

$$x' = \frac{x}{L}, \quad \xi_1' = \frac{\xi_1}{L}, \quad \xi_2' = \frac{\xi_2}{L}, \quad t' = \sqrt{\frac{g}{L}}t, \; T' = \sqrt{\frac{g}{L}}T, \quad u_1 = -\frac{m_1\ddot{\xi}_1}{Mg}, \quad u_2 = \frac{M_2\ddot{\xi}_2}{Mg}$$

$$f = \frac{F}{Mg}, \quad U_1 = \frac{m_1A_1}{Mg}, \quad U_2 = \frac{m_2A_2}{Mg} \tag{1.11}$$

where L is an arbitrary parameter that has the dimension of length.

In what follows, the primes denoting dimensionless variables are omitted and the dot stands for a derivative with respect to the dimensionless time t'.

We will seek the motions of the internal bodies which maximize the average velocity of the system for fixed T or, which is the same, the displacement of the main body draining this period.

The quantities u_1 and u_2 are treated as the control variables. Since u_i is proportional to the second derivative of the periodic function ξ_i, the functions u_i have zero means over the period T. Conversely, if this condition holds, the periodic functions $\xi_i(t)$, which satisfy the condition $\xi_i(0) = 0$, are uniquely found in the form

$$\xi_i(t) = (-1)^i\frac{M}{m_i}\left[\frac{t}{T}\int_0^T \tau u_i(\tau)d\tau + \int_0^t (t-\tau)u_i(\tau)d\tau\right], \quad i = 1, 2 \tag{1.12}$$

Thus we arrive at the optimal control problem.

Problem 1. For the system

$$\ddot{x} = u_1 + f$$

$$f = \begin{cases} -k(1+u_2)\operatorname{sign}\dot{x}, & \text{if } \dot{x} \neq 0 \\ -u_1, & \text{if } \dot{x} = 0 \text{ and } |u_1| \leq k(1+u_2) \\ -k(1+u_2)\operatorname{sign}(u_1), & \text{if } \dot{x} = 0 \text{ and } |u_1| > k(1+u_2) \end{cases} \tag{1.13}$$

subject to the boundary conditions

$$x(0) = 0, \quad \dot{x}(0) = \dot{x}(T) = 0 \tag{1.14}$$

it is required to find controls $u_1(t)$ and $u_2(t)$ that satisfy the constraints

$$\int_0^T u_i(t)dt = 0, \quad i = 1, 2 \tag{1.15}$$

$$|u_1(t)| \leq U_1, \quad -U_2^- \leq u_2(t) \leq U_2 \tag{1.16}$$

where

$$U_2^- = \min\{U_2, 1\}, \quad U_1 > k(1 - U_2^-) \tag{1.17}$$

and maximize the quantity $x(T)$:

$$x(T) \to \max_{u_1, u_2}$$

Relations (1.13) were obtained from (1.1)–(1.3), conditions (1.14) from (1.8), and relations (1.16) and (1.17) correspond to (1.9) and (1.10).

2. Properties of the optimal motion

In this section, a number of propositions concerning the properties of the optimal motion and the structure of the optimal control law for Problem 1 are proved. The controls u_i that satisfy constraints (1.15) and (1.16) and the condition $\dot{x}(T) = 0$, if $\dot{x}(0) = 0$, will be referred to as admissible controls.

Proposition 1. *For the optimal controls, the main body of the system moves forward or remains in a state of rest but never moves backward*:

$$\dot{x} \geq 0, \quad t \in [0, T] \tag{2.1}$$

Proof. Assume the contrary. Let $u_1(t)$ and $u_2(t)$ be admissible controls for which the velocity $\dot{x}$ of the main body is negative ($\dot{x} < 0$) in some interval $(t_1, t_2) \subset [0, T]$ and vanishes at the ends of this interval: $\dot{x}(t_1) = \dot{x}(t_2) = 0$. From relations (1.13) and the inequality $\dot{x} < 0$ it follows that the motion of the main body in the interval (t_1, t_2) is governed by the equation

$$\ddot{x} = u_1 + k(1 + u_2)$$

Integrating this equation in the interval $[t_1, t_2]$ and taking the conditions $\dot{x}(t_1) = \dot{x}(t_2) = 0$ into account we obtain the following relations between the average values $\bar{u}_i$ of the functions u_i:

$$\bar{u}_1 + k(1 + \bar{u}_2) = 0, \quad \bar{u}_i = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} u_i(\tau)d\tau, \quad i = 1, 2$$

We introduce the new control functions

$$\tilde{u}_i(t) = \begin{cases} u_i(t), & t \in [0, T]\backslash(t_1, t_2) \\ \bar{u}_i, & t \in (t_1, t_2) \end{cases}, \quad i = 1, 2$$

These functions satisfy the constraints (1.15) and (1.16), since these constraints are satisfied by the original controls $u_1(t)$ and $u_2(t)$. □

For the new controls, the main body of the system remains at rest in the time interval $[t_1, t_2]$, while in the remaining portion of the interval $[0, T]$ it moves with a velocity identical with that corresponding to the original controls. Since, by assumption, the velocity of the main body is negative in the interval (t_1, t_2) for the original controls and is equal to zero for the new controls, the new controls provide a larger value of the variable $x(T)$. Hence, the original control is non-optimal, and for the optimal control, the velocity of the main body is non-negative in the entire interval $[0, T]$.

Proposition 2. *Let $u_s(t)$ be admissible controls such that $\dot{x} \geq 0$ for $t \in [0, T]$. Then admissible controls $\hat{u}_s(t)$ and some $\delta \in [0, T]$ exist for which the following relations are satisfied*

$$\dot{x} > 0 \text{ almost everywhere for } t \in [0, \delta]; \quad \dot{x} \equiv 0 \text{ for } t \in [\delta, T] \tag{2.2}$$

and drive the main body at the instant T to the same position as the controls $u_s(t)$ do.

Proof. The controls $u_s(t)$ generate the representation of the half-open interval $[0, T)$ as the sum of non-intersecting half-open intervals $\Delta_\alpha^k = [t_\alpha^k, \tilde{t}_\alpha^k)$, where $\alpha = +, 0$ for the intervals in which $\dot{x} > 0$ almost everywhere and $\dot{x} \equiv 0$, respectively:

$$[0, T] = \cup_{i,j}(\Delta_+^i \cup \Delta_0^j)$$

At the ends of the intervals Δ_+^i, the velocity $\dot{x}$ vanishes. Let δ denote the total length of the intervals Δ_+^i. Consider the piecewise linear one-to-one mapping of the interval $[0, T]$ onto itself

$$\phi\colon [0, T] \to [0, T]; \quad \phi(t) = t + c_\alpha^k, \quad t \in \Delta_\alpha^k$$

where the constants c_α^k are chosen so as to satisfy the condition

$$\phi(\Delta_+^i) \in [0, \delta], \quad \phi(\Delta_0^j) \in [\delta, T]$$

Define the controls $\hat{u}_s(t)$ as follows:

$$\hat{u}_s(t) = u_s(\phi^{-1}(t))$$

Together with this relation, the mapping $\phi(t)$ "shuffles" the time intervals together with the control laws defined in these intervals. The displacements of the main body are the same for the intervals Δ_α^k and $\phi(\Delta_\alpha^k)$ and, hence, the final positions of the main body for the controls $\hat{u}_s$ and $u_s(t)$ coincide. By construction, the controls $\hat{u}_s(t)$ satisfy relations (2.2). This completes the proof of Proposition 2. □

Propositions 1 and 2 enable the class of admissible controls to be restricted to the functions $u_s(t)$ that generate the motions for which relations (2.2) are satisfied.

Proposition 3. *Let u_1^* and u_2^* be the optimal controls that solve Problem 1 and such that $\dot{x} > 0$ almost everywhere in $[0, \delta^*]$ and $\dot{x} \equiv 0$ in $[\delta^*, T]$. Then the optimal controls in the interval $[0, \delta^*]$ have the form*

$$u_1^* = \begin{cases} U_1, & t \in [0, \tau_1) \\ -U_1, & t \in [\tau_1, \delta^*] \end{cases} \tag{2.3}$$

$$u_2^* = \begin{cases} -U_2^-, & t \in [0, \tau_2) \\ U_2, & t \in [\tau_2, \delta^*] \end{cases} \tag{2.4}$$

Proof. Let the optimal control $u_2^*(t)$ be fixed in the interval $[0, T]$. Consider controls u_1 that have the form

$$u_1 = \begin{cases} u^1(t), & t \in [0, \delta^*] \\ u_1^*(t), & t \in [\delta^*, T] \end{cases}$$

satisfy the constraints of Problem 1 and ensure that the relations $\dot{x}(\delta^*) = 0$ and $\dot{x} > 0$ are satisfied almost everywhere in the interval $(0, \delta^*)$, subject to the initial condition $\dot{x}(0) = 0$. □

The controls u_1 under consideration ensure that the relation $x(\delta^*) = x(T)$ is satisfied . Therefore, the control u_1^*, $t \in [0, \delta^*]$, is a solution of the following optimal control problem: for the system

$$\ddot{x} = u^1 - k(1 + u_2^*) \tag{2.5}$$

subject to the boundary conditions

$$x(0) = 0, \quad \dot{x}(0) = \dot{x}(\delta^*) = 0 \tag{2.6}$$

it is required to find the control $u^1(t)$ that satisfies the constraints

$$|u^1(t)| \le U_1 \tag{2.7}$$

$$\int_0^{\delta^*} u^1(t)dt = -\int_{\delta^*}^{T} u_1^*(t)dt \tag{2.8}$$

and maximizes $x(\delta^*)$, i.e.,

$$x(\delta^*) \to \max \tag{2.9}$$

From Eq. (2.5) and boundary conditions (2.6) it follows that

$$\int_0^{\delta^*} u^1(t)dt = \int_0^{\delta^*} u_1^*(t)dt = k\int_0^{\delta^*} (1 + u_2^*(t))dt$$

Hence, condition (2.8) always holds.

We apply the maximum principle to problem (2.5)–(2.7) and (2.9) to prove that the optimal control $u_1^*(t)$ has the form (2.3). In a similar manner, one can prove that the optimal control $u_2^*(t)$ has the form (2.4).

3. Construction of the optimal control

In accordance with the above observations, we will seek the optimal controls u_i^* in the class of functions

$$u_1(t) = \begin{cases} U_1, & t \in [0, \tau_1) \\ -U_1, & t \in [\tau_1, \delta), \\ f_1, & t \in [\delta, T] \end{cases} \quad u_2(t) = \begin{cases} -U_2^-, & t \in [0, \tau_2) \\ U_2, & t \in [\tau_2, \delta) \\ f_2, & t \in [\delta, T] \end{cases} \tag{3.1}$$

These functions must satisfy the conditions of the optimal control problem and the corresponding motion of the main body must be characterized by the relations $\dot{x}(0) = 0$ and $\dot{x} > 0$ almost everywhere for $t \in (0, \delta)$, and $\dot{x} \equiv 0$ for $t \in [\delta, T]$.

The control functions $f_i(t)$ in the interval $[\delta, T]$ must satisfy the constraints

$$|f_1| \le U_1, \quad -U_2^- \le f_2 \le U_2 \tag{3.2}$$

$$\int_\delta^T f_i(t)dt = -\int_0^\delta u_i(t)dt \tag{3.3}$$

and the inequality

$$|f_1| \le k(1 + f_2) \tag{3.4}$$

which ensures rest of the main body, $\dot{x} \equiv 0$, in the interval $[\delta, T]$, provided that $\dot{x}(\delta) = 0$. If relations (3.2)–(3.4) hold for the functions $f_i(t)$, they also hold for the averages of these functions. Therefore, we can confine ourselves to the class of constant functions f_i.

Thus, the controls u_i are specified parametrically and the solution of the optimal control problem can be reduced to the determination of the optimal values of the parameters τ_1, τ_2, δ, f_1, and f_2, maximizing the quantity $x(T)$. We will derive the conditions which must be satisfied for these parameters.

The motion of the main body in the interval $[0, \delta]$ is described by the equation

$$\ddot{x} = u_1 - k(1 + u_2) \tag{3.5}$$

subject to the boundary conditions $\dot{x}(0) = \dot{x}(\delta) = 0$, by virtue of which the following integral relation is satisfied

$$\int_0^\delta u_1(t)dt - k\delta - k\int_0^\delta u_2(t)dt = 0$$

Substituting the control law of (3.1) into the last relation we obtain the following relation between the parameters τ_1, τ_2, and δ:

$$\tau_1 = \frac{\delta[U_1 + k(1 + U_2)] - 2k\tilde{U}_2\tau_2}{2U_1}; \quad \tilde{U}_2 = \frac{U_2^- + U_2}{2} \tag{3.6}$$

Expressing the parameters f_1 and f_2 in terms of τ_2 and δ, using relation (3.3), the control law (3.1), and the relation between the parameters (3.6), we obtain

$$f_1 = -\frac{k[\delta(1 + U_2) - 2\tilde{U}_2\tau_2]}{T - \delta}, \quad f_2 = \frac{2\tilde{U}_2\tau_2 - U_2\delta}{T - \delta} \tag{3.7}$$

Inequalities (3.2) and (3.4) for f_i are transformed into inequalities for τ_2 and δ

$$-U_1(T - \delta) \le k(\delta(1 + U_2) - 2\tilde{U}_2\tau_2) \le U_1(T - \delta) \tag{3.8}$$

$$-U_2^-(T - \delta) \le 2\tilde{U}_2\tau_2 - U_2\delta \le U_2(T - \delta) \tag{3.9}$$

$$|\delta(1 + U_2) - 2\tilde{U}_2\tau_2| \le T + 2\tilde{U}_2\tau_2 - (1 + U_2)\delta \tag{3.10}$$

These inequalities, combined with the conditions $\tau_1 \in [0, \delta]$, $\tau_2 \in [0, \delta]$, and $\delta \in [0, T]$ define the admissible domain for the parameters τ_2 and δ. The system of inequalities for this domain can be reduced to the form

$$\delta[U_1 + k(1 + U_2)] \le U_1T + 2k\tilde{U}_2\tau_2 \tag{3.11}$$

$$2\tilde{U}_2\delta - U_2^-T \le 2\tilde{U}_2\tau_2 \le U_2T \tag{3.12}$$

$$(1 + U_2)\delta \le \frac{T}{2} + 2\tilde{U}_2\tau_2 \tag{3.13}$$

$$\max\left\{0, \frac{\delta[k(1+U_2)-U_1]}{2k\tilde{U}_2}\right\} \le \tau_2 \le \delta \le T \tag{3.14}$$

Relation (3.11) is the right-hand inequality of (3.8). The left-hand inequality of (3.8) holds, since $\tau_2 \le \delta \le T$ and $U_2^- \le 1$. Relation (3.12) is a transform of inequality (3.9). Relation (3.13) is equivalent to inequality (3.10) for $T \ge 0$.

We will calculate the displacement of the main body of system $x(T)$ as a function of the parameters τ_2 and δ. Since $\dot{x} \equiv 0$ in the interval $[\delta, T]$, we have the equality $x(\delta) = x(T)$. Solving Eq. (3.5), governing the motion of the main body, subject to the controls (3.1) and the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$, taking into account relation (3.6) between the parameters τ_i, we obtain

$$x(T) = \frac{U_1^2 - k^2(1+U_2)^2}{4U_1}\delta^2 + k\tilde{U}_2\frac{U_1 + k(1+U_2)}{U_1}\tau_2\delta - k\tilde{U}_2\frac{U_1 + k\tilde{U}_2}{U_1}\tau_2^2 \tag{3.15}$$

Hence, the solution of the optimal control problem is reduced to maximizing the function (3.15), quadratic in τ_2 and δ, in the convex polygon of admissible values of τ_2 and δ, defined by inequalities (3.11)–(3.14). Provided condition (3.14) is satisfied, the quantity $x(T)$ increases monotonically as δ increases in the domain of admissible values of τ_2 and δ, and $x(T)$ reaches a maximum on the boundary of this domain.

4. Limiting cases

We will calculate the maximum value of the average velocity of motion of the main body

$$V = x(T)/T$$

in two limiting cases, $U_2 = \infty$ and $U_2 = 0$. In the first case, the function $u_2(t)$, characterizing the control of the motion of the internal mass in the vertical direction, has no upper limit. In the second case, $u_2(t) \equiv 0$ and, hence, there is no motion of the internal mass along the vertical.

Case 1. $U_2 = \infty$. In this case, $U_2^- = 1$ in accordance with the first relation of (1.17). From the system of inequalities (3.11)–(3.14) it follows that $\tau_2 \to \delta$ as $U_2 \to \infty$. For any $\tau_2 = \delta$ and $\delta \in [0, T]$, this system of inequalities is satisfied in the limit as $U_2 \to \infty$. Substituting $\tau_2 = \delta = T$ into expression (3.15) we obtain

$$x(T) = U_1T^2/4 \tag{4.1}$$

We will show that this solution is optimal. To that end, consider the following auxiliary optimal control problem.

Problem 2. For the equation $\ddot{x} = u_1 + f$, subject to conditions (1.14), it is required to find the controls $u_1(t)$ and $f(t)$ that satisfy conditions (1.15), (1.16), and

$$\dot{x}f \le 0, \quad \text{if} \quad \dot{x} \ne 0; \quad u_1 f \le 0, \quad \text{if} \quad \dot{x} = 0 \tag{4.2}$$

and maximize the quantity $x(T)$:

$$x(T) \to \max_{u_1, f}$$

The quantity f in Problem 2 can be regarded as the dry friction force generated by the normal pressure force, the magnitude of which ranges from 0 to ∞ and is treated as the control function. Thus, this problem deals with the system controlled by the internal mass along the horizontal and by an external force along the vertical. Problem 2 can be obtained from Problem 1 for $U_2 = \infty$ by omitting the integral constraint of (1.15) for u_2. Therefore, the maximum value of $x(T)$ in Problem 2 is not less than the maximum value of $x(T)$ in Problem 1 for $U_2 = \infty$.

Proposition 4. *The optimal control in Problem 2 has the form*

$$u_1(t) = \begin{cases} U_1, & t \in [0, T/2) \\ -U_1, & t \in [T/2, T) \end{cases}, \quad f \equiv 0$$

and the corresponding value of the performance index is given by expression (4.1).

Proof. Propositions 1–3, proved above for Problem 1, remain valid for Problem 2. Since $\dot{x} > 0$and, hence, $f \le 0$, almost everywhere in the interval $[0, \delta]$, maximum of $x(\delta)$ for fixed $u_1(t)$ is reached for $f \equiv 0$ in the interval $t \in [0, \delta)$. □

We will find the optimal control $u_1(t)$ in the interval $[0, \delta]$ for $f \equiv 0$. According to Proposition 3, $u_1 = U_1\text{sign}(\tau_1 - t)$. From constraint (1.15) for u_1 it follows that $\tau_1 \le T/2$. By choosing τ_1 so as to maximize $x(\delta)$ we obtain

$$\tau_1 = \begin{cases} \delta & \text{for} \quad \delta \le T/2 \\ T/2 & \text{for} \quad \delta > T/2 \end{cases}$$

$$x(\delta) = \begin{cases} U_1\delta^2/2 & \text{for} \quad \delta \le T/2 \\ U_1\delta^2/2 - U_1(\delta - T/2)^2 & \text{for} \quad \delta > T/2 \end{cases}$$

The integral relation of (1.15) for u_1 can be satisfied by letting $u_1 = U_1 \operatorname{sign}(T/2 - t)$ for $t \in [0, T]$. The state of rest of the main body for $t \in (T/2, T]$ can be ensured by an appropriate choice of the function f, since the conditions of Problem 2 allow the main body to be decelerated instantaneously to a complete stop and to be kept in a state of rest indefinitely due to the choice of this function. Since the function $x(\delta)$ increases monotonically, the optimal control corresponds to $\delta = T$. This completes the proof of Proposition 4.

This proposition implies the optimality of the solution $\tau_2 = \delta = T$ and $x(T) = U_1T^2/4$ for the limiting case $U_2 = \infty$ of Problem 1. From relation (3.6) we find $\tau_1 = T/2$. Thus, the optimal control parameters τ_1, τ_2, and δ are given by

$$\tau_1 = T/2, \quad \tau_2 = T, \quad \delta = T$$

and the interval $[\delta, T]$, in which the main body remains at rest, degenerates to the point $t = T$. The optimal control u_1 has the form

$$u_1(t) = \begin{cases} U_1, & t \in [0, T/2) \\ -U_1, & t \in [T/2, T) \end{cases} \tag{4.3}$$

In the half-open interval $[0, T)$, the optimal control u_2 is defined as $u_2 = -U_2^- = -1$.

According to the second relation of (3.7), in the case when $U_2^- = 1$ we obtain

$$f_2 = \frac{U_2(\tau_2 - \delta) + \tau_2}{T - \delta}$$

For $\tau_2 = \delta$ and $\delta \to T$, we have

$$f_2 \sim \frac{\delta}{T - \delta} \to \infty, \quad \int_\delta^T f_2(t)dt \to T$$

This means that the optimal control $u_2(t)$ contains on impulse component of intensity T, represented by the Dirac delta function $T\delta(t - T)$, concentrated at the instant of time T and, finally, has the form

$$u_2(t) = -1 + T\delta(t - T) \tag{4.4}$$

For this control, the normal pressure force exerted by the main body on the plane is zero in the half-open interval $[0, T)$ and, hence, the friction force is zero in this interval, i.e., $f \equiv 0$.

The optimal motion of the main body is described by the relations

$$x(t) = \begin{cases} U_1t^2/2, & t \in [0, T/2) \\ U_1[T^2/4 - (T - t)^2/2], & t \in [T/2, T] \end{cases}$$
$$\dot{x}(t) = \begin{cases} U_1t, & t \in [0, T/2) \\ U_1(T - t), & t \in [T/2, T] \end{cases} \tag{4.5}$$

$$V = U_1T/4 \tag{4.6}$$

This motion, being periodically extended to the infinite time interval $[0, \infty)$, is velocity stable. Specifically, if the velocity of the main body is non-zero at the initial instant of time, i.e., $\dot{x}(0) = \dot{x}_0 \neq 0$, and the system is subjected to the optimal controls of (4.3) and (4.4), the body will reach the velocity-periodic motion mode of (4.5) in a finite time.

In fact, the integration of Eq. (1.13) gives

$$\dot{x}(T) = \dot{x}_0 + \int_0^T u_1(\tau)d\tau + \int_0^T f(\tau)d\tau$$

Since $f(t) \equiv 0$ for $t \in [0, T)$ and the second term on the right-hand side of this relation is equal to zero in accordance with condition (1.15), the velocity of the main body at the instant $t = T - 0$, just before the action of the impulse component of the control u_2, is equal to its initial value: $x(T - 0) = x_0$. The impulse component of the control u_2, in accordance with expression (1.13) for f, generates an impulse of the friction force that instantaneously reduces the magnitude of the velocity of the main body by an amount kT, if $kT < |\dot{x}_0|$, or brings it to a complete stop, if $kT \ge |\dot{x}_0|$. Therefore,

$$\dot{x}(T) = \begin{cases} (|\dot{x}_0| - kT)\operatorname{sign}(\dot{x}_0), & \text{if} \quad |\dot{x}_0| \le kT \\ 0, & \text{if} \quad |\dot{x}_0| \le kT \end{cases}$$

and, hence,

$$\dot{x}(t_*) = 0, \quad t_* = \left(\operatorname{int}\left(\frac{|\dot{x}_0|}{kT}\right) + 1\right)T$$

where the function int denotes the integer part of its argument. Consequently, from the instant t_*, the motion follows the velocity-periodic mode (4.5).

We will obtain the motion of the internal masses corresponding to controls (4.3) and (4.4). From relations (1.12) we obtain

$$\xi_1 = \frac{MU_1}{4m_1} \times \begin{cases} t(T-2t), & 0 \le t \le T/2 \\ 2t^2 - 3Tt + T^2, & T/2 \le t \le T \end{cases}$$

$$\xi_2 = \frac{M}{2m_2}t(T-t), \quad 0 \le t \le T$$

Then the vibration amplitudes of the internal masses are expressed as

$$\Delta\xi_1 = \frac{MU_1T^2}{16m_1}, \quad \Delta\xi_2 = \frac{MT^2}{8m_2}; \quad \Delta\xi_i = \max_{t \in [0,T)} \xi_i(t) - \min_{t \in [0,T)} \xi_i(t) \tag{4.7}$$

Let the vibration amplitudes of the internal masses be constrained as follows:

$$\Delta\xi_1 \le L_1, \quad \Delta\xi_2 \le L_2 \tag{4.8}$$

We will vary the period T to maximize the average velocity V for these constraints on the amplitudes. By solving relations (4.7) and (4.8) for T we obtain an expression for the maximum possible period and then, using expression (4.6), we find V:

$$T = 4\sqrt{\frac{m_1L_1}{MU_1}}\min\{1, \kappa\}, \quad V = \sqrt{\frac{m_1U_1L_1}{M}}\min\{1, \kappa\}; \quad \kappa = \sqrt{\frac{m_2U_1L_2}{2m_1L_1}}$$

For any specified values of the maximum amplitudes L_1 and L_2 allowed for the vibration of the internal masses, the quantity V increases without limit as $U_1 \to \infty$. Hence, an arbitrarily high average velocity can be ensured in principle for the system, provided the drives are powerful enough.

Case 2. $U_2 = 0$. In this case, which has been studied in detail previously,[9] there is no motion of the internal mass along the vertical. Taking into account the inequality $U_1 > k$, implied by the second relation of (1.17), relations (3.11)–(3.14) reduce to the inequality $\tau_2 \le \delta \le T/2$, while the switching instant τ_1 and the displacement of the main body during the period are calculated from the formulae

$$\tau_1 = \frac{\delta}{2}\left(1 + \frac{k}{U_1}\right), \quad x(T) = \frac{U_1^2 - k^2}{4U_1}\delta^2$$

From these relations, the optimal parameters of the control law, the maximum displacement of the main body during the period, and the average velocity of the optimal motion are defined by the expressions

$$\delta = \frac{T}{2}, \quad \tau_1 = \tau^* = \frac{T}{4}\left(1 + \frac{k}{U_1}\right), \quad x(T) = VT, \quad V = \frac{U_1T}{16}\left[1 - \frac{k^2}{U_1^2}\right] \tag{4.9}$$

In the first half of the period, the main body moves forward, speeding up with an acceleration of $U_1 - k$ in the interval $[0, \tau^*]$ and slowing down with an acceleration of $-U_1 - k$ in the interval $[\tau^*, T/2]$; in the second half of the period, the main body remains at rest.

The average velocity of the system has been calculated[9] for the control law cited above, with the period T at which the vibration amplitude of the internal mass is equal to the prescribed value L_1. It has been shown that this velocity tends to infinity as the maximum acceleration U_1 allowed for the internal mass increases without limit. Moreover, the stability of the velocity of the main body in the optimal motion with respect to the initial perturbation of this velocity has been proved. Unlike the case when $U_2 = \infty$, the perturbed velocity reaches the optimal mode in a finite time only when $\dot{x}_0 > 0$, while when $\dot{x}_0 < 0$, the convergence is exponential.

The ratio of expression (4.6) to the last expression of (4.9) for the average velocities of the system for fixed T in the cases $U_2 = \infty$ and $U_2 = 0$ equals $\frac{4U_1^2}{U_1^2 - k^2}$. Therefore, for given total mass of the system and the constraints on the relative acceleration of the internal mass moving horizontally, the control of the normal pressure by means of the motion of the internal mass along the vertical makes it possible in principle to obtain at least a 4-fold increase in the average velocity of the motion, as compared with the maximum average velocity of motion of a system in which there is no internal mass moving along the vertical.

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